

Coulomb scattering in the Born approximation and the use of generalized functions: Supplementary material

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I. GENERALIZED FUNCTIONS

In this Section we present the minimum amount of the theory of generalized functions required in order to put the calculations in Sec. IV of the main paper on a mathematically sound basis. In addition, we believe that this presentation will make it easier for readers to pursue the subject further in any of the excellent textbooks [1], [2], [3], [4], [5]. For a review of generalized functions in connection with applications to electromagnetism we recommend [6].

We shall be concerned with applications involving Fourier transforms and for that reason we shall require a set of “test” functions $\varphi(x)$ (initially in R^1) which are called *good or rapidly decreasing test functions*.

Definition 1 A function φ is said to be good if $\varphi \in C^\infty$ and if

$$\lim_{|x| \rightarrow \infty} \left| x^m \frac{d^k}{dx^k} \varphi(x) \right| = 0, \quad (1)$$

for every integer $m \geq 0$ and every integer $k \geq 0$. It is evident that if φ is good so is $d\varphi/dx$.

In the customary notation we write, $\varphi \in S$, where S is the space of good test functions.

Now we let f be a functional. The functional f assigns a number to any good test function φ denoted $\langle f, \varphi \rangle$. If f is a locally integrable function, then we may write

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x) \varphi(x) dx. \quad (2)$$

However if f is not a locally integrable function, then the right-hand side of Eq. (2) does not make sense. In order for $\langle f, \varphi \rangle$ to be a finite number for $\varphi \in S$, it is sufficient, *but not necessary*, to restrict $f(x)$ to the space of functions of slow growth. This is an important set of functions that will enable us to deal with cases where the right-hand side of Eq. (2) does not make sense, and give meaning to $\langle f, \varphi \rangle$ by introducing the concept of *generalized functions*, (see Example 1 below). The terms *generalized function* and *distribution* will be used interchangeably in what follows.

Definition 2 A function $f(x)$ is said to be a function of slow growth if, for some (finite) integer $N \geq 0$,

$$\int_{-\infty}^{\infty} \frac{|f(x)|}{(1+x^2)^N} dx < \infty. \quad (3)$$

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We say that $f \in K$. The generalization of Definition 2 to n dimensions is straightforward, $x^2 \rightarrow r^2 = x_1^2 + \cdots + x_n^2$, etc. For example (for $n = 3$) the function $f = (1/r) \in K$ (use spherical coordinates). Good functions decrease faster than any power of $|x|$ as $x \rightarrow \pm\infty$, e.g., $\exp(-x^2)$, (but note that $\exp(-|x|)$ is not a good function). Functions f of slow growth grow at infinity like polynomials, e.g., $\exp(ix)$.

Remark 1 *An important and rather obvious consequence is a theorem that states that the product of a function f of slow growth and a rapidly decreasing function φ , is a rapidly decreasing function $f\varphi$.*

Definition 3 *The piecewise continuous function of slow growth $f(x)$, defines the tempered distribution (or distribution of slow growth)*

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx, \quad (4)$$

for all good functions φ .

The set of all tempered distributions is denoted by S' .

We can now differentiate $f(x)$ using integration by parts,

$$\int_{-\infty}^{\infty} \frac{df(x)}{dx} \varphi(x)dx = [f(x)\varphi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \frac{d\varphi(x)}{dx} dx. \quad (5)$$

It follows from Remark 1 that $[f(x)\varphi(x)]_{-\infty}^{\infty} = 0$, therefore we simply have that

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle, \quad (6)$$

$$\langle f'', \varphi \rangle = -\langle f', \varphi' \rangle = \langle f, \varphi'' \rangle, \quad (7)$$

and so forth. Note that Eqs. (6) and (7) hold even if $f(x)$ is not differentiable (since φ is).

Remark 2 *It is easy to see that the generalized derivative, Eq. (6), of a generalized function, f , is also a generalized function, f' .*

Example 1 *As our first example we show how to obtain the Dirac δ function as the derivative of the unit step function*

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases} \quad (8)$$

Although the derivative, $\theta'(0)$ does not exist in the usual sense. $\theta(x) \in K$ and $\varphi \in S$, therefore from Eq. (6), we have

$$\langle \theta', \varphi \rangle = - \int_{-\infty}^{\infty} \theta(x) \frac{d\varphi}{dx} dx = - \int_0^{\infty} \frac{d\varphi}{dx} dx \quad (9)$$

$$= -\varphi(x)|_0^{\infty} = \varphi(0) := \langle \delta, \varphi \rangle \in S'. \quad (10)$$

So $\theta'(x)$ maps every test function $\varphi(x)$ to its value at the origin and enables us to define the generalized function $\delta(x)$.

We now turn our attention to the Fourier transform. We recall that the Fourier transform $\widehat{f}(x)$ (or $[f(x)]^\wedge$), of a well-behaved function $f(x)$, is

$$\widehat{f}(q) = \int_{-\infty}^{\infty} f(x) e^{-iqx} dx. \quad (11)$$

We need to make use of *Parseval's equation*, which for well-behaved functions f and g , is easy to prove using Fubini's theorem [?],

$$\int_{-\infty}^{\infty} \widehat{f}(q) g(q) dq = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{-iqx} dx \right] g(q) dq \quad (12)$$

$$= \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} g(q) e^{-iqx} dq \right] dx \quad (13)$$

$$= \int_{-\infty}^{\infty} f(x) \widehat{g}(x) dx. \quad (14)$$

Definition 4 Let $f(x)$ be a piecewise continuous function of slow growth, then we use Parseval's equation to define the Fourier transform \widehat{f} , of the generalized function f , to be

$$\langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle, \quad (15)$$

for all good functions φ .

In order for Definition 4 to be meaningful we need the theorem below which we state without proof.

Theorem 1 If $\varphi(x)$ is a good function so is its Fourier transform $\widehat{\varphi}(q)$.

This is the reason why the test functions φ had to be good. From Theorem 1 and Definition 3 it follows that if f is a function of slow growth, then its Fourier transform \widehat{f} is a tempered distribution (generalized function of slow growth).

Since φ is a good function, it is easy to deduce by direct calculation that for k a non-negative integer

$$\partial^k (\widehat{\varphi}) = [(-ix)^k \varphi]^\wedge, \quad (16)$$

$$[\partial^k \varphi]^\wedge = (iq)^k \widehat{\varphi}. \quad (17)$$

Example 2 *We show that the Fourier transform of the δ function is 1.*

$$\langle \widehat{\delta}, \varphi \rangle = \langle \delta, \widehat{\varphi} \rangle = \widehat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(x) dx = \langle 1, \varphi \rangle, \quad (18)$$

where we have used Eq. (10).

Remark 3 *The above example is simple but it is important to note that every step in Eqs (18) is independent of the particular φ .*

Using Eqs. (14), (6) and (7) we obtain two very handy results.

Corollary 1

$$\langle \widehat{(f')}, \varphi \rangle = \langle f', \widehat{\varphi} \rangle = -\langle f, (\widehat{\varphi})' \rangle. \quad (19)$$

Corollary 2

$$\langle \widehat{(f'')}, \varphi \rangle = \langle f'', \widehat{\varphi} \rangle = -\langle f', (\widehat{\varphi})' \rangle = \langle f, (\widehat{\varphi})'' \rangle. \quad (20)$$

We re-write Eq. (20) for clarity

$$\int_{-\infty}^{\infty} \left[\frac{d^2}{dx^2} f(x) \right]^\wedge (q) \varphi(q) dq = \int_{-\infty}^{\infty} f(x) \left(\frac{d^2}{dx^2} \widehat{\varphi}(x) \right) dx. \quad (21)$$

Remark 4 *All of the preceding results in this Section can be extended (with appropriate minor changes) to several dimensions, that is, $x \in R^n$. Thus, for example, for $x \in R^n$ Eqs. (7) and (20) become*

$$\langle \Delta f, \varphi \rangle = \langle f, \Delta \varphi \rangle, \quad (22)$$

$$\langle \widehat{(\Delta f)}, \varphi \rangle = \langle f, \Delta (\widehat{\varphi}) \rangle, \quad (23)$$

where

$$\Delta = (\partial_1^2 + \cdots + \partial_n^2). \quad (24)$$

The first step in the derivation of Eq. (29) of the main paper, is to prove the following theorem.

Theorem 2

$$\int_{R^3} (\Delta f) \varphi d^3r = \int_{R^3} \frac{f}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) d^3r, \quad (25)$$

where $f = f(r) \in K$, (a function of slow growth), $\varphi = \varphi(r, \theta_1, \theta_2) \in S$, (a good function), and $d^3r := r^2 dr d\Omega = r^2 dr \sin \theta_1 d\theta_1 d\theta_2$.

Proof:

Remark 5 In ref. [1] it is shown that in R^n it is sufficient to work with good functions which are the product of n good functions φ_k , each of which is a function of a single variable.

Thus in R^3 we may write, without loss of generality,

$$\varphi = \varphi(r, \theta_1, \theta_2) = \varphi(x, y, z) = \varphi_1(x) \varphi_2(y) \varphi_3(z) \quad (26)$$

$$= \varphi_1(r \sin \theta_1 \cos \theta_2) \varphi_2(r \sin \theta_1 \sin \theta_2) \varphi_3(r \cos \theta_1). \quad (27)$$

We shall make use of the fact that φ is periodic in θ_2 , and

$$\lim_{r \rightarrow 0} \varphi(r, \theta_1, \theta_2) := \varphi(0). \quad (28)$$

From Eq. (22) we have that

$$\int_{R^3} (\Delta f) \varphi d^3r = \int_{R^3} f \Delta \varphi d^3r, \quad (29)$$

and, in spherical coordinates,

$$\Delta \varphi = A + B + C, \quad (30)$$

where

$$A = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) \quad (31)$$

$$B = \frac{1}{r^2 \sin \theta_1} \frac{\partial}{\partial \theta_1} \left(\sin \theta_1 \frac{\partial \varphi}{\partial \theta_1} \right) \quad (32)$$

$$C = \frac{1}{r^2 \sin^2 \theta_1} \frac{\partial^2 \varphi}{\partial \theta_2^2}. \quad (33)$$

Thus we write

$$\int_{R^3} f \Delta \varphi d^3 r = \int_{R^3} f(A + B + C) r^2 dr d\Omega, \quad (34)$$

and consider the third term on the right-hand side,

$$\int_{R^3} f C r^2 dr d\Omega = \int \dots \int_0^{2\pi} \frac{\partial^2 \varphi}{\partial \theta_2^2} d\theta_2 = \int \dots \left[\frac{\partial \varphi}{\partial \theta_2} \right]_0^{2\pi} = 0, \quad (35)$$

because φ has period 2π in θ_2 , (see Eq. (27)). Now we consider the second term on the right-hand side of Eq. (34),

$$\int_{R^3} f B r^2 dr d\Omega = \int \dots \int_0^\pi \frac{\partial}{\partial \theta_1} \left(\sin \theta_1 \frac{\partial \varphi}{\partial \theta_1} \right) d\theta_1 \quad (36)$$

$$= \int \dots \left[\sin \theta_1 \frac{\partial \varphi}{\partial \theta_1} \right]_0^\pi = 0, \quad (37)$$

since $\partial \varphi / \partial \theta_1$ is bounded and continuous. Therefore,

$$\int_{R^3} (\Delta f) \varphi d^3 r = \int_{R^3} f A d^3 r, \quad (38)$$

which completes the proof of Eq. (25).

We now proceed to the final step required to prove Eq. (29) of the main paper. We let

$$f = \frac{1}{r}, \quad \Rightarrow \quad \frac{\partial f}{\partial r} = -\frac{1}{r^2}. \quad (39)$$

Recall that $f \in K$. Then integrating by parts Eq. (38), we have

$$\int_{R^3} (\Delta f) \varphi d^3 r = \int_{R^3} f A d^3 r \quad (40)$$

$$= \int_{R^3} f \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) dr d\Omega \quad (41)$$

$$= \int d\Omega \left[f r^2 \frac{\partial \varphi}{\partial r} \right]_0^\infty - \int_{R^3} \frac{\partial f}{\partial r} r^2 \frac{\partial \varphi}{\partial r} d^3 r \quad (42)$$

$$= \int_{R^3} \frac{\partial \varphi}{\partial r} dr d\Omega = \int_0^{2\pi} \int_0^\pi [\varphi]_0^\infty d\Omega \quad (43)$$

$$= -\varphi(0) \int_0^{2\pi} \int_0^\pi d\Omega = -4\pi \varphi(0), \quad (44)$$

where we have used Eq. (28). Using the generalization of Eq. (10), we have shown that

$$\int_{R^3} (\Delta f) \varphi d^3r = -4\pi \int_{R^3} \delta(\vec{r}) \varphi(\vec{r}) d^3r = -4\pi \varphi(0). \quad (45)$$

It is in this sense that we may write

$$\Delta \left(\frac{1}{r} \right) = -4\pi \delta(\vec{r}). \quad (46)$$

Remark 6 *We believe that the following statement about $\delta(x)$ from Friedman's early text [7] captures the essence of generalized functions: "We notice that the function $\delta(\vec{r})$ is treated exactly as if it were an ordinary function except that we shall never talk about the "values" of $\delta(\vec{r})$. We talk about the values of integrals involving $\delta(\vec{r})$ ".*

We do need to prove one last proposition, namely, Eq. (4.4) of the main paper.

Proposition 1 *If $f \in K$, then*

$$[\Delta f]^\wedge(\vec{q}) = -q^2 \widehat{f}(\vec{q}). \quad (47)$$

Proof:

We shall make use of the generalization of Eq. (16) to R^3 , namely,

$$\Delta \widehat{\varphi} = [-q^2 \varphi]^\wedge. \quad (48)$$

Thus

$$\langle \widehat{\Delta f}, \varphi \rangle = \langle \Delta f, \widehat{\varphi} \rangle = \langle f, \Delta \widehat{\varphi} \rangle \quad (49)$$

$$= \langle f, [-q^2 \varphi]^\wedge \rangle = \langle \widehat{f}, -q^2 \varphi \rangle \quad (50)$$

$$= \langle -q^2 \widehat{f}, \varphi \rangle, \quad (51)$$

which is Eq. (47), or Eq. (24) of the main paper.

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